

Conservations of first-order reflections

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Abstract

The set theory $K\Pi_{N+1}$ for Π_{N+1} -reflecting universes is shown to be Π_{N+1} -conservative over iterations of Π_N -recursively Mahlo operations for each $N \geq 2$.

1 Introduction

It is well known that the set of weakly Mahlo cardinals below a weakly compact cardinal is stationary. Furthermore any weakly compact cardinal κ is in the diagonal intersection $\kappa \in M^\Delta = \bigcap \{M(M^\alpha) : \alpha < \kappa\}$ for the α -th iterate M^α of the Mahlo operation M , where $\kappa \in M(X)$ iff $X \cap \kappa$ is stationary in κ .

The same holds for the recursive analogues of the indescribable cardinals, *reflecting ordinals* introduced by Richter-Aczel [12]. First let us recall the ordinals briefly. For a full account of the admissible set theory, see [8].

Δ_0 denotes the set of bounded formulas in the language $\{\in\}$ of set theories. Then the classes Σ_{i+1}, Π_{i+1} are defined recursively as usual. Each class Σ_{i+1}, Π_{i+1} is defined to be closed under bounded quantifications $\exists x \in a, \forall x \in a$.

The axioms of the Kripke-Platek set theory with the axiom of infinity, denoted $KP\omega$, are Extensionality, Foundation schema, Pair, Union, Δ_0 -Separation, Δ_0 -Collection, and the axiom of infinity. Note that except Foundation schema, each axiom in $KP\omega$ is a Π_2 -formula.

For set-theoretic formulas φ , let $P \models \varphi :\Leftrightarrow (P, \in) \models \varphi$.

In what follows, let V denote a transitive and wellfounded model of $KP\omega$, which is a universe in discourse. P, Q, \dots denote non-empty transitive sets in $V \cup \{V\}$.

A Π_i -recursively Mahlo operation for $2 \leq i < \omega$, is defined through a universal Π_i -formula $\Pi_i(a)$:

$$P \in RM_i(\mathcal{X}) \quad :\Leftrightarrow \quad \forall b \in P [P \models \Pi_i(b) \rightarrow \exists Q \in \mathcal{X} \cap P (b \in Q \models \Pi_i(b))]$$

(read: P is Π_i -reflecting on \mathcal{X} .)

For the universe V , $V \in RM_i(\mathcal{X})$ denotes $\forall b [\Pi_i(b) \rightarrow \exists Q \in \mathcal{X} (b \in Q \models \Pi_i(b))]$. Suppose that there exists a first-order sentence φ such that $P \in \mathcal{X} \Leftrightarrow P \models \varphi$

for any transitive $P \in V \cup \{V\}$. Then $RM_i(\mathcal{X})$ is Π_{i+1} , i.e., there exists a Π_{i+1} -sentence $rm_i(\mathcal{X})$ such that $P \in RM_i(\mathcal{X})$ iff $P \models rm_i(\mathcal{X})$ for any transitive set P .

The iteration of RM_i along a definable relation \prec is defined as follows.

$$P \in RM_i(a; \prec) :\Leftrightarrow a \in P \in \bigcap \{RM_i(RM_i(b; \prec)) : b \in P \models b \prec a\}.$$

Again $P \in RM_i(a; \prec)$ is a Π_{i+1} -relation.

Let Ord denote the class of ordinals in V . Let us write RM_i^α for $RM_i(\alpha; <)$ and ordinals $\alpha \in Ord$. A transitive set P is said to be Π_i -reflecting if $P \in RM_i = RM_i^1$.

$P \in RM_{i+1}$ is much stronger than $P \in RM_i$: Assume $P \in RM_{i+1}$ and $P \models \Pi_i(b)$ for $b \in P$. Then $P \in RM_i$ and $P \models rm_i \wedge \Pi_i(b)$ for the Π_{i+1} -sentence rm_i such that $P \in RM_i$ iff $P \models rm_i$. Hence there exists a $Q \in P$ such that $Q \models rm_i \wedge \Pi_i(b)$, i.e., $Q \in RM_i$ & $Q \models \Pi_i(b)$. This means $P \in RM_i^2 = RM_i(RM_i)$. Moreover P is in the diagonal intersection of RM_i , $P \in RM_i^\Delta$, i.e., $P \in \bigcap \{RM_i^\beta : \beta \in P \cap Ord\}$, and so on.

In particular a set theory $K\Pi_{i+1}$ for universes in RM_{i+1} proves the consistency of a set theory for universes in RM_i^Δ .

In this paper we address a problem: How far can we iterate lower recursively Mahlo operations in higher reflecting universes? In [1] we gave a sketchy proof of the following Theorem 1.1, which is implicit in ordinal analyses in [2, 4].

Theorem 1.1 *For each $N \geq 2$ there exists a Σ_1 -relation \triangleleft_N on ω such that the set theory KPl for limits of admissibles proves the transfinite induction schema for \triangleleft_N up to each $a \in \omega$, and $K\Pi_{N+1}$ is Π_1^1 (on ω)-conservative over the theory*

$$KPl + \{V \in RM_N(a; \triangleleft_N) : a \in \omega\}.$$

Theorem 1.1 suffices to approximate $K\Pi_{N+1}$ proof-theoretically in terms of iterations of Π_N -recursively Mahlo operations. However $V \in RM_N(a; \prec)$ is a Π_{N+1} -formula for Σ_{N+1} -relation \prec , and the class Π_1^1 on ω is smaller than Π_{N+1} .

In this paper the set theory $K\Pi_{N+1}$ for Π_{N+1} -reflecting universes is shown to be Π_{N+1} -conservative over iterations of Π_N -recursively Mahlo operations RM_N for each $N \geq 2$. This result will be extended in [6, 7] to the indescribable cardinals over $ZF + (V = L)$.

2 Conservation

Let $Ord \subset V$ denote the class of ordinals, $Ord^\varepsilon \subset V$ and $<^\varepsilon$ be Δ -predicates such that for any transitive and wellfounded model V of $KP\omega$, $<^\varepsilon$ is a well ordering of type $\varepsilon_{\Omega+1}$ on Ord^ε for the order type Ω of the class Ord in V . Specifically let us encode ‘ordinals’ $\alpha < \varepsilon_{\Omega+1}$ by codes $[\alpha] \in Ord^\varepsilon$ as follows. $[\alpha] = \langle 0, \alpha \rangle$ for $\alpha \in Ord$, $[\Omega] = \langle 1, 0 \rangle$, $[\omega^\alpha] = \langle 2, [\alpha] \rangle$ for $\alpha > \Omega$, and $[\alpha] = \langle 3, [\alpha_1], \dots, [\alpha_n] \rangle$ if $\alpha = \alpha_1 + \dots + \alpha_n > \Omega$ with $\alpha_1 \geq \dots \geq \alpha_n$, $n > 1$

and $\exists \beta_i (\alpha_i = \omega^{\beta_i})$ for each α_i . Then $[\omega_n(\Omega + 1)] \in \text{Ord}^\varepsilon$ denotes the code of the ‘ordinal’ $\omega_n(\Omega + 1)$.

$<^\varepsilon$ is assumed to be a canonical ordering such that $\text{KP}\omega$ proves the fact that $<^\varepsilon$ is a linear ordering, and for any formula φ and each $n < \omega$,

$$\text{KP}\omega \vdash \forall x (\forall y <^\varepsilon x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x <^\varepsilon [\omega_n(\Omega + 1)] \varphi(x) \quad (1)$$

For a definition of Δ -predicates Ord^ε and $<^\varepsilon$, and a proof of (1), cf. [5].

Proposition 2.1 *$\text{KP}\omega$ proves that if $P \in \text{RM}_N(\beta; <^\varepsilon)$, then $\forall \alpha <^\varepsilon \beta (\alpha \in P \rightarrow P \in \text{RM}_N(\alpha; <^\varepsilon))$.*

Proof. This is seen from the fact that $<^\varepsilon$ is transitive in $\text{KP}\omega$. \square

Theorem 2.2 *For each $N \geq 2$, KPII_{N+1} is Π_{N+1} -conservative over the theory*

$$\text{KP}\omega + \{V \in \text{RM}_N([\omega_n(\Omega + 1)]; <^\varepsilon) : n \in \omega\}.$$

From (1) we see that KPII_{N+1} proves $V \in \text{RM}_N([\omega_n(\Omega + 1)]; <^\varepsilon)$ for each $n \in \omega$. The converse is proved in section 3.

Proposition 2.3 *For any class Γ of Π_{N+1} -sentences, there exists a Σ_{N+1} -sentence A such that $\text{KPII}_{N+1} \vdash A$, and $\text{KP}\omega + \Gamma \not\vdash A$ unless $\text{KP}\omega + \Gamma$ is inconsistent.*

Proof. This follows from the essential unboundedness theorem due to G. Kreisel and A. Lévy [10]. In this proof let $\vdash A :\Leftrightarrow \text{KP}\omega \vdash A$ and Pr denote a standard provability predicate for $\text{KP}\omega$. Also $\text{Tr}_{\Pi_{N+1}}$ denotes a partial truth definition of Π_{N+1} -sentences.

Then let A be a Σ_{N+1} -sentence saying that ‘I am not provable from any true Π_{N+1} -sentence’, $\vdash A \leftrightarrow \forall x \in \omega [\text{Tr}_{\Pi_{N+1}}(x) \rightarrow \neg \text{Pr}(x \dot{\rightarrow} [A])]$, where $\dot{\rightarrow}$ denotes a recursive function such that $[A] \dot{\rightarrow} [B] = [A \rightarrow B]$ for codes $[A]$ of formulas A .

Suppose $\text{KP}\omega + \Gamma \vdash A$. Pick a $C \in \Gamma$ so that $\vdash C \rightarrow A$. Then $\text{KP}\omega + \Gamma \vdash \text{Tr}_{\Pi_{N+1}}([C]) \wedge \text{Pr}([C \rightarrow A])$. Hence $\text{KP}\omega + \Gamma \vdash \neg A$.

In what follows argue in KPII_{N+1} . Suppose A is false, and let C be any true Π_{N+1} -sentence. Since the universe V is Π_{N+1} -reflecting, there exists a transitive model $P \in V$ of $\text{KP}\omega + \{C, \neg A\}$, which shows that $\text{KP}\omega + \{C, \neg A\}$ is consistent. In other words, $\neg \text{Pr}([C \rightarrow A])$. Therefore $\text{KPII}_{N+1} \vdash \neg A \rightarrow A$. \square

Thus Theorem 2.2 is optimal with respect to the class Π_{N+1} of formulas provided that KPII_{N+1} is consistent.

Corollary 2.4 *For each $N \geq 3$, $\text{KPII}_{N+1} + (\text{Power}) + (\Sigma_{N-2}\text{-Separation}) + (\Pi_{N-2}\text{-Collection})$ is Π_{N+1} -conservative over the theory $\text{KP}\omega + \{V \in \text{RM}_N([\omega_n(\Omega + 1)]; <^\varepsilon) : n \in \omega\} + (\text{Power}) + (\Sigma_{N-2}\text{-Separation}) + (\Pi_{N-2}\text{-Collection})$.*

Proof. This follows from Theorem 2.2 and the facts that the axiom Power is a Π_3 -sentence $\forall a \exists b \forall x \subset a(x \in b)$, and Σ_i -Separation or Π_i -Collection are Π_{i+2} -formulas. \square

Let us announce an extension of Theorem 2.2 in [6, 7] to the indescribable cardinals over $\mathbf{ZF} + (V = L)$.

Let $<^\varepsilon$ be an ε -ordering as above. Let M_N denote the Π_N^1 -Mahlo operation defined for sets S of ordinals and uncountable regular cardinals κ : $\kappa \in M_N(S)$ iff $S \cap \kappa$ is Π_N^1 -indescribable in κ . The Π_{N+1}^1 -indescribability is proof-theoretically reducible to iterations of an operation along initial segments of $<^\varepsilon$ over $\mathbf{ZF} + (V = L)$. The operation is a mixture of the Π_N^1 -Mahlo operation M_N and Mostowski collapsings.

For $\alpha <^\varepsilon \varepsilon_{\mathcal{K}+1}$ and finite sets $\Theta \subset_{fin} (\mathcal{K} + 1)$, Π_{n+1} -classes $Mh_n^\alpha[\Theta]$ are defined so that the following holds.

In Theorem 2.5 \mathcal{K} is intended to denote the least Π_{N+1}^1 -indescribable cardinal, and Ω the least weakly inaccessible cardinal above \mathcal{K} .

Theorem 2.5 (The case $N = 0$ in [6], and the general case in [7].)

1. For each $n < \omega$,

$$\mathbf{ZF} + (V = L) + (\mathcal{K} \text{ is } \Pi_{N+1}^1\text{-indescribable}) \vdash \mathcal{K} \in Mh_n^{\omega_n(\Omega+1)}[\emptyset].$$

2. For any Σ_{N+2}^1 -sentences φ , if

$$\mathbf{ZF} + (V = L) + (\mathcal{K} \text{ is } \Pi_{N+1}^1\text{-indescribable}) \vdash \varphi^{L\mathcal{K}},$$

then we can find an $n < \omega$ such that

$$\mathbf{ZF} + (V = L) + (\mathcal{K} \in Mh_n^{\omega_n(\Omega+1)}[\emptyset]) \vdash \varphi^{L\mathcal{K}}.$$

The classes $Mh_n^\alpha[\Theta]$ are defined from iterated Skolem hulls $\mathcal{H}_{\alpha,n}(X)$, through which we described the limit of $\mathbf{ZF} + (V = L)$ -provable countable ordinals in [5] as follows.

Theorem 2.6 ([5].)

$$\begin{aligned} |\mathbf{ZF} + (V = L)|_{\omega_1} &:= \inf\{\alpha \leq \omega_1 : \forall \varphi [\mathbf{ZF} + (V = L) \vdash \exists x \in L_{\omega_1} \varphi \Rightarrow \exists x \in L_\alpha \varphi]\} \\ &= \Psi_{\omega_1 \varepsilon_{\Omega+1}} := \sup\{\Psi_{\omega_1, n} \omega_n(\Omega + 1) : n < \omega\}. \end{aligned}$$

In Theorem 2.6, Ω is intended to denote the least weakly inaccessible cardinal.

3 Proof of Theorem 2.2

In this section we prove Theorem 2.2. Our proof is extracted from M. Rathjen's ordinal analyses of Π_3 -reflection in [11].

Let $N \geq 2$ denote a fixed integer. The axioms of the set theory KPII_{N+1} for Π_{N+1} -reflecting universes are those of $\text{KP}\omega$ and the axiom for Π_{N+1} -reflection: for Π_{N+1} -formulas φ , $\varphi(a) \rightarrow \exists c[ad^c \wedge a \in c \wedge \varphi^c(a)]$, where ad denotes a Π_3 -sentence such that $P \models ad$ iff P is a transitive model of $\text{KP}\omega$, and φ^c denotes the result of restricting any unbounded quantifiers $\exists x, \forall x$ in φ to $\exists x \in c, \forall x \in c$, resp.

KPi denotes the set theory for recursively inaccessible sets, which is obtained from $\text{KP}\omega$ by adding the axiom $\forall x \exists y[x \in y \wedge ad^y]$.

Throughout this section we work in an intuitionistic fixed point theory $\text{FiX}^i(\text{KPi})$ over KPi . The intuitionistic theory $\text{FiX}^i(\text{KPi})$ is introduced in [5], and shown to be a conservative extension of KPi . Let us reproduce definitions and results on $\text{FiX}^i(\text{KPi})$ here.

Fix an X -strictly positive formula $\mathcal{Q}(X, x)$ in the language $\{\in, =, X\}$ with an extra unary predicate symbol X . In $\mathcal{Q}(X, x)$ the predicate symbol X occurs only strictly positive. This means that the predicate symbol X does not occur in the antecedent φ of implications $\varphi \rightarrow \psi$ nor in the scope of negations \neg in $\mathcal{Q}(X, x)$. The language of $\text{FiX}^i(\text{KPi})$ is $\{\in, =, Q\}$ with a fresh unary predicate symbol Q . The axioms in $\text{FiX}^i(\text{KPi})$ consist of the following:

1. All provable sentences in KPi (in the language $\{\in, =\}$).
2. Induction schema for any formula φ in $\{\in, =, Q\}$:

$$\forall x(\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x) \quad (2)$$

3. Fixed point axiom:

$$\forall x[Q(x) \leftrightarrow \mathcal{Q}(Q, x)].$$

The underlying logic in $\text{FiX}^i(\text{KPi})$ is defined to be the intuitionistic (first-order predicate) logic (with equality).

(2) yields the following Lemma 3.1.

Lemma 3.1 *Let $<^\varepsilon$ denote a Δ_1 -predicate mentioned in the beginning of section 2. For each $n < \omega$ and each formula φ in $\{\in, =, Q\}$,*

$$\text{FiX}^i(\text{KPi}) \vdash \forall x(\forall y <^\varepsilon x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x <^\varepsilon \omega_n(I+1)\varphi(x).$$

The following Theorem 3.2 is seen as in [3, 5].

Theorem 3.2 *$\text{FiX}^i(\text{KPi})$ is a conservative extension of KPi .*

In what follows we work in $\text{FiX}^i(\text{KPi})$.

Let V denote a transitive and wellfounded model of $\text{KP}\omega$. Consider the language $\mathcal{L}_V = \{\in\} \cup \{c_a : a \in V\}$ where c_a denotes the name of the set $a \in V$. We identify the set a with its name c_a .

Our proof proceeds as follows. Assume that $\text{KPII}_{N+1} \vdash A$ for a Π_{N+1} -sentence A . KPII_{N+1} is embedded to an infinitary system R_N formulated in one-sided sequent calculus, and cut inferences are eliminated, which results in

an infinitary derivation of height $\alpha < \varepsilon_{\Omega+1}$ with an inference rule (Ref_{N+1}) for Π_{N+1} -reflection. Then A is seen to be true in $P \in RM_N(\alpha; <^\varepsilon)$.

In one-sided sequent calculi, formulas are generated from atomic formulas and their negations $a \in b, a \notin b$ by propositional connectives \vee, \wedge and quantifiers \exists, \forall . It is convenient here to have bounded quantifications $\exists x \in a, \forall x \in a$ besides unbounded ones $\exists x, \forall x$. The negation $\neg A$ of formulas A is defined recursively by de Morgan's law and elimination of double negations. Also $(A \rightarrow B) := (\neg A \vee B)$.

Γ, Δ, \dots denote finite sets of sentences, called *sequents* in the language \mathcal{L}_V . Γ, Δ denotes the union $\Gamma \cup \Delta$, and Γ, A the union $\Gamma \cup \{A\}$. A finite set Γ of sentences is intended to denote the disjunction $\bigvee \Gamma := \bigvee \{A : A \in \Gamma\}$. Γ is *true* in $P \in V \cup \{V\}$ iff $\bigvee \Gamma$ is true in P iff $\bigvee \Gamma^P$ is true.

Classes $\Delta_0, \Sigma_{i+1}, \Pi_{i+1}$ of sentences in \mathcal{L}_V are defined as usual.

We assign disjunctions or conjunctions to sentences as follows. When a disjunction $\bigvee (A_\iota)_{\iota \in J}$ [a conjunction $\bigwedge (A_\iota)_{\iota \in J}$] is assigned to A , we denote $A \simeq \bigvee (A_\iota)_{\iota \in J}$ [$A \simeq \bigwedge (A_\iota)_{\iota \in J}$], resp.

Definition 3.3 1. For a Δ_0 -sentence M

$$M := \begin{cases} \bigvee (A_\iota)_{\iota \in J} & \text{if } M \text{ is false in } V \\ \bigwedge (A_\iota)_{\iota \in J} & \text{if } M \text{ is true in } V \end{cases} \quad \text{with } J := \emptyset.$$

In what follows consider the unbounded sentences.

2. $(A_0 \vee A_1) \simeq \bigvee (A_\iota)_{\iota \in J}$ and $(A_0 \wedge A_1) \simeq \bigwedge (A_\iota)_{\iota \in J}$ with $J := 2$.
3. $\exists x \in a A(x) \simeq \bigvee (A(b))_{b \in J}$ and $\forall x \in a A(x) \simeq \bigwedge (A(b))_{b \in J}$ with $J := a$.
4. $\exists x A(x) \simeq \bigvee (A(b))_{b \in J}$ and $\forall x A(x) \simeq \bigwedge (A(b))_{b \in J}$ with $J := V$.

Definition 3.4 The *depth* $\text{dp}(A) < \omega$ of \mathcal{L}_V -sentences A is defined recursively as follows.

1. $\text{dp}(A) = 0$ if $A \in \Delta_0$.

In what follows consider unbounded sentences A .

2. $\text{dp}(A) = \max\{\text{dp}(A_i) : i < 2\} + 1$ if $A \equiv (A_0 \circ A_1)$ for $\circ \in \{\vee, \wedge\}$.
3. $\text{dp}(A) = \text{dp}(B(\emptyset)) + 1$ if $A \in \{(Qx B(x)), (Qx \in a B(x)) : a \in V\}$ for $Q \in \{\exists, \forall\}$.

Definition 3.5 1. For \mathcal{L}_V -sentences A , $\mathbf{k}(A) := \{a \in V : c_a \text{ occurs in } A\}$.

2. For sets Γ of sentences, $\mathbf{k}(\Gamma) := \bigcup \{\mathbf{k}(A) : A \in \Gamma\}$.

3. For $\iota \in V$ and a transitive model $P \in V \cup \{V\}$ of $\text{KP}\omega$, $P(\iota) \in V \cup \{V\}$ denotes the smallest transitive model of $\text{KP}\omega$ such that $P \cup \{\iota\} \subset P(\iota)$, cf. [8].

For finite lists $\vec{a} = (a_1, \dots, a_n)$, $P(\vec{a}) := (\dots P(a_1) \dots)(a_n)$.

Inspired by operator controlled derivations due to W. Buchholz [9], let us define a relation $P \vdash_m^\alpha \Gamma$ for transitive models $P \in V \cup \{V\}$ of $\text{KP}\omega$. The relation $P \vdash_m^\alpha \Gamma$ is defined as a fixed point of a strictly positive formula H

$$H(P, \alpha, m, \Gamma) \Leftrightarrow P \vdash_m^\alpha \Gamma$$

in $\text{FiX}^i(\text{KP}i)$.

Note that P contains the code $\langle 1, 0 \rangle = [\Omega]$, and is closed under ordinal addition $(\alpha, \beta) \mapsto \alpha + \beta$, exponentiation $\alpha \mapsto \omega^\alpha$ for $\alpha, \beta \in \text{Ord}^\varepsilon$ and $a \mapsto \text{rank}(a)$ for $\text{rank}(a) = \sup\{\text{rank}(b) + 1 : b \in a\}$.

Definition 3.6 Let $P \in V \cup \{V\}$ be a transitive model of $\text{KP}\omega$, $\alpha < \varepsilon_{\Omega+1}$ and $m < \omega$.

$P \vdash_m^\alpha \Gamma$ holds if

$$\mathbf{k}(\Gamma) \cup \{\alpha\} \subset P \quad (3)$$

and one of the following cases holds:

(\bigvee) there is an $A \in \Gamma$ such that $A \simeq \bigvee (A_\iota)_{\iota \in J}$, and for an $\iota \in J$ and an $\alpha(\iota) < \alpha$, $P \vdash_m^{\alpha(\iota)} \Gamma, A_\iota$.

$$\frac{P \vdash_m^{\alpha(\iota)} \Gamma, A_\iota}{P \vdash_m^\alpha \Gamma} (\bigvee)$$

(\bigwedge) there is an $A \in \Gamma$ such that $A \simeq \bigwedge (A_\iota)_{\iota \in J}$, and for any $\iota \in J$, there is an $\alpha(\iota)$ such that $\alpha(\iota) < \alpha$ and $P \vdash_m^{\alpha(\iota)} \Gamma, A_\iota$.

$$\frac{\{P \vdash_m^{\alpha(\iota)} \Gamma, A_\iota : \iota \in J\}}{P \vdash_m^\alpha \Gamma} (\bigwedge)$$

(*cut*) there are C and α_0, α_1 such that $\text{dp}(C) < m$, $\alpha_0, \alpha_1 < \alpha$, and $P \vdash_m^{\alpha_0} \Gamma, \neg C$ and $P \vdash_m^{\alpha_1} C, \Gamma$.

$$\frac{P \vdash_m^{\alpha_0} \Gamma, \neg C \quad P \vdash_m^{\alpha_1} C, \Gamma}{P \vdash_m^\alpha \Gamma} (\text{cut})$$

(Ref_{N+1}) there are $A(c) \in \Pi_{N+1}$ and $\alpha_0, \alpha_1 < \alpha$ such that $P \vdash_m^{\alpha_0} \Gamma, A(c)$ and $P \vdash_m^{\alpha_1} \forall z[ad^z \rightarrow c \in z \rightarrow \neg A^z], \Gamma$.

$$\frac{P \vdash_m^{\alpha_0} \Gamma, A(c) \quad P \vdash_m^{\alpha_1} \forall z[ad^z \rightarrow c \in z \rightarrow \neg A^z], \Gamma}{P \vdash_m^\alpha \Gamma} (\text{Ref}_{N+1})$$

In what follows, let us fix an integer n_0 and restrict (codes of) ordinals to $\alpha <^\varepsilon [\omega_{n_0}(\Omega + 1)]$. n_0 is chosen from the given finite proof of a Π_{N+1} -sentence A in KPII_{N+1} , cf. Corollary 3.9 (Embedding). Since n_0 is a constant, we see from Lemma 3.1 that $\text{FiX}^i(\text{KP}i)$ proves transfinite induction schema up to $[\omega_{n_0}(\Omega + 1)]$ for any formula in which the derivability relation $P \vdash_m^\alpha \Gamma$ may occur.

Proposition 3.7 *Let $P' \supset P$ be a transitive model of $\text{KP}\omega$, $m \leq m' < \omega$ and $\text{k}(\Delta) \cup \{\alpha'\} \subset P'$. If $P \vdash_m^\alpha \Gamma$, then $P' \vdash_{m'}^{\alpha'} \Gamma, \Delta$.*

In embedding KPII_{N+1} in the infinitary calculus, it is convenient to formulate KPII_{N+1} in (finitary) one-sided sequent calculus of the language $\{\in, 0\}$ with the individual constant 0 for the empty set. Axioms are logical ones $\Gamma, \neg A, A$ for any formulas A , and axioms in the theory KPII_{N+1} . Inference rules are (\vee) , (\wedge) for propositional connectives, $(b\exists)$, $(b\forall)$ for bounded quantifications, (\exists) , (\forall) for unbounded quantifications, and (cut) . For details, see the proof of the following Lemma 3.8.

Though the following Lemmas 3.8, 3.10 and 3.11 are seen as in [9], we give proofs of them for readers' convenience.

Let $(m, \vec{a}) := \Omega \cdot m + 3\text{rank}(a_1) \# \cdots \# 3\text{rank}(a_n)$ for $\vec{a} = (a_1, \dots, a_n)$ and the natural (commutative) sum $\alpha \# \beta$ of ordinals α, β .

Lemma 3.8 *Suppose $\text{KPII}_{N+1} \vdash \Gamma(\vec{x})$, where free variables occurring in the sequent are among the list \vec{x} . Then there is an $m < \omega$ such that for any $\vec{a} \subset V$ and any transitive model $P \in V \cup \{V\}$ of $\text{KP}\omega$, $P(\vec{a}) \vdash_m^{(m, \vec{a})} \Gamma(\vec{a})$.*

Proof. First consider the logical axiom $\Gamma(\vec{x}), \neg A(\vec{x}), A(\vec{x})$. We see that for any \vec{a}

$$P(\vec{a}) \vdash_0^{2d} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a}) \quad (4)$$

by induction on $d = \text{dp}(A)$.

Then by Proposition 3.7 we have $P(\vec{a}) \vdash_{2d}^{(2d, \vec{a})} \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a})$.

If $d = 0$, then $A \in \Delta_0$ and one of $\neg A(\vec{a})$ and $A(\vec{a})$ is true. Hence by (\wedge) we have $P(\vec{a}) \vdash_0^0 \Gamma(\vec{a}), \neg A(\vec{a}), A(\vec{a})$.

Next consider the case when $A \equiv (\exists y B(\vec{x}, y)) \notin \Delta_0$ with $\text{dp}(B(\vec{x}, y)) = d - 1$. By IH(=Induction Hypothesis) we have for any $\vec{a} \subset V$ and any $b \in V$, $P(\vec{a} * (b)) \vdash_0^{2d-2} \Gamma(\vec{a}), \neg B(\vec{a}, b), B(\vec{a}, b)$, where $(a_1, \dots, a_n) * (b) = (a_1, \dots, a_n, b)$. (\vee) yields $P(\vec{a} * (b)) \vdash_0^{2d-1} \Gamma(\vec{a}), \neg B(\vec{a}, b), \exists y B(\vec{a}, y)$. Hence (\wedge) with $P(\vec{a} * (b)) = P(\vec{a})(b)$ yields $P(\vec{a}) \vdash_0^{2d} \Gamma(\vec{a}), \neg \exists y B(\vec{a}, y), \exists y B(\vec{a}, y)$.

The cases $A \equiv (\exists y \in a B(\vec{x}, y)) \notin \Delta_0$ and $A \equiv (B_0 \vee B_1) \notin \Delta_0$ are similar. Thus (4) was shown.

Second consider the inference rule (\exists) with $\exists y A(\vec{x}, y) \in \Gamma(\vec{x})$

$$\frac{\Gamma(\vec{x}), A(\vec{x}, t)}{\Gamma(\vec{x})} (\exists)$$

When t is a variable y , we can assume that y is an x_i in the list \vec{x} , for otherwise substitute 0 for y . By IH there is an m such that $P(\vec{a}) \vdash_m^{(m, \vec{a})} \Gamma(\vec{a}), A(\vec{a}, t')$ where $t' \equiv a_i$ if $t \equiv x_i$, and $t' \equiv 0$ otherwise. Thus $P(\vec{a}) \vdash_{m+1}^{(m+1, \vec{a})} \Gamma(\vec{a})$.

Third consider the inference rule (\forall) with $\forall y A(\vec{x}, y) \in \Gamma(\vec{x})$

$$\frac{\Gamma(\vec{x}), A(\vec{x}, y)}{\Gamma(\vec{x})} (\forall)$$

where the variable y does not occur in $\Gamma(\vec{x})$. IH yields for an m , $P(\vec{a}*(b)) \vdash_m^{(m, \vec{a}*(b))} \Gamma(\vec{a}), A(\vec{a}, b)$. (\wedge) with $(m+1, \vec{a}) > (m, \vec{a}*(b))$ yields $P(\vec{a}) \vdash_{m+1}^{(m+1, \vec{a})} \Gamma(\vec{a})$.

The following cases are similarly seen.

$$\frac{\Gamma, t \in s \quad \Gamma, B(\vec{x}, t)}{\Gamma, \exists y \in s B(\vec{x}, y)} (b\exists) \quad \frac{\Gamma, y \notin s, B(\vec{x}, y)}{\Gamma, \forall y \in s B(\vec{x}, y)} (b\forall) \quad \frac{\Gamma, A_0, A_1}{\Gamma, A_0 \vee A_1} (\vee) \quad \frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \wedge A_1} (\wedge)$$

In a cut inference

$$\frac{\Gamma(\vec{x}), \neg A(\vec{x}) \quad A(\vec{x}), \Gamma(\vec{x})}{\Gamma(\vec{x})} (cut)$$

if the cut formula $A(\vec{x})$ has free variables \vec{y} other than \vec{x} , then substitute 0 for \vec{y} .

In what follows let us suppress parameters.

Fourth consider the axioms other than Foundation. For example, consider the Δ_0 -Collection $\forall x \in a \exists y A(x, y) \rightarrow \exists z \forall x \in a \exists y \in z A(x, y)$ for $A \in \Delta_0$ and $a \in V$. Since $P(a)$ is a transitive model of $KP\omega$ and $a \in P(a)$, pick a $b \in P(a)$ such that $\forall x \in a \exists y A(x, y) \rightarrow \forall x \in a \exists y \in b A(x, y)$ holds in $P(a)$. Then $\neg \forall x \in a \exists y A(x, y) \vee \forall x \in a \exists y \in b A(x, y)$ is a true Δ_0 -sentence. Hence $P(a) \vdash_0^0 \neg \forall x \in a \exists y A(x, y), \forall x \in a \exists y \in b A(x, y)$. Three (\vee) 's yield $P(a) \vdash_0^3 \forall x \in a \exists y A(x, y) \rightarrow \exists z \forall x \in a \exists y \in z A(x, y)$.

Next consider the axiom $A(c) \rightarrow \exists z[ad^z \wedge c \in z \wedge A^z]$ for $A \in \Pi_{N+1}$. We have by (4) for $d = \text{dp}(A)$

$$\frac{P(c) \vdash_0^{2d} \neg A(c), A(c) \quad P(c) \vdash_0^2 \forall z[ad^z \rightarrow c \in z \rightarrow \neg A^z], \exists z[ad^z \wedge c \in z \wedge A^z]}{P(c) \vdash_0^{2d+1} \neg A(c), \exists z[ad^z \wedge c \in z \wedge A^z]} (Ref_{N+1})$$

In this way we see that there are cut-free infinitary derivations of finite heights deducing axioms in $K\Pi_{N+1}$ other than Foundation.

Finally consider the Foundation. Let $d = \text{dp}(A)$ and $B \equiv (\neg \forall x (\forall y \in x A(y) \rightarrow A(x)))$. We show by induction on $\text{rank}(a)$ that

$$P(a) \vdash_0^{2d+3\text{rank}(a)} B, \forall x \in a A(x) \quad (5)$$

By IH we have for any $b \in a$, $P(b) \vdash_0^{2d+3\text{rank}(b)} B, \forall x \in b A(x)$. Thus we have by (4)

$$\frac{P(b) \vdash_0^{2d+3\text{rank}(b)} B, \forall x \in b A(x) \quad P(b) \vdash_0^{2d} \neg A(b), A(b)}{P(b) \vdash_0^{2d+3\text{rank}(b)+1} B, \forall x \in b A(x) \wedge \neg A(b), A(b)} (\wedge) \\ \frac{P(b) \vdash_0^{2d+3\text{rank}(b)+1} B, \forall x \in b A(x) \wedge \neg A(b), A(b)}{P(b) \vdash_0^{2d+3\text{rank}(b)+2} B, A(b)} (\vee)$$

Therefore (5) was shown.

$$\frac{\{P(a, b) \vdash_0^{2d+3\text{rank}(b)+2} B, A(b) : b \in a\}}{P(a) \vdash_0^{2d+3\text{rank}(a)} B, \forall x \in a A(x)} (\wedge)$$

□

Corollary 3.9 (Embedding) *If $\text{KPII}_{N+1} \vdash A$ for a sentence A , then there is an $m < \omega$ such that for any transitive model $P \in V \cup \{V\}$ of $\text{KP}\omega$, $P \vdash_m^{\Omega \cdot m} A$.*

Lemma 3.10 (Reduction) *Let $C \simeq \bigvee (C_\iota)_{\iota \in J}$. Then*

$$(P \vdash_m^\alpha \Delta, \neg C) \& (P \vdash_m^\beta C, \Gamma) \& (\text{dp}(C) \leq m) \Rightarrow P \vdash_m^{\alpha+\beta} \Delta, \Gamma$$

Proof. This is seen by induction on β .

Consider first the case when C is a Δ_0 -sentence. Then C is false and $J = \emptyset$. From $P \vdash_m^\beta C, \Gamma$ we see that $P \vdash_m^\beta \Gamma$. $\beta \leq \alpha + \beta$ yields $P \vdash_m^{\alpha+\beta} \Delta, \Gamma$.

Next assume that the last inference rule in $P \vdash_m^\beta C, \Gamma$ is a (\bigvee) with the main formula $C \notin \Delta_0$:

$$\frac{P \vdash_m^{\beta(\iota)} C, C_\iota, \Gamma}{P \vdash_m^\beta C, \Gamma} (\bigvee)$$

where $\iota \in J$ and $\beta(\iota) < \beta$. We can assume that ι occurs in C_ι . Otherwise set $\iota = 0$. Thus $\iota \in P$ by (3). On the other hand we have $P(\iota) \vdash_m^\alpha \Delta, \neg C_\iota$ by inversion, and hence $P \vdash_m^\alpha \Delta, \neg C_\iota$ by $\iota \in P$.

IH yields $P \vdash_m^{\alpha+\beta(\iota)} C_\iota, \Delta, \Gamma$. A cut inference with $P \vdash_m^\alpha \Delta, \neg C_\iota$ and $\text{dp}(C_\iota) < \text{dp}(C) \leq m$ yields $P \vdash_m^{\alpha+\beta} \Delta, \Gamma$. \square

Lemma 3.11 (Predicative Cut-elimination) $P \vdash_{m+1}^\alpha \Gamma \Rightarrow P \vdash_m^{\omega^\alpha} \Gamma$.

Proof. This is seen by induction on α using Reduction 3.10 and the fact: $\beta < \alpha \Rightarrow \omega^\beta + \omega^\beta \leq \omega^\alpha$. \square

For $\alpha <^\varepsilon [\omega_n(\Omega + 1)]$, set $RM_N^\alpha := RM_N(\alpha; <^\varepsilon)$.

Proposition 3.12 *Let $\Gamma \subset \Pi_{N+1}$ and $P \in RM_N^\alpha$ be a transitive model of KPi . Assume*

$$\exists \xi, x \in P (0 <^\varepsilon \xi <^\varepsilon \alpha \wedge \forall Q \in RM_N^\xi \cap P (x \in Q \models \text{KPi} \rightarrow \Gamma \text{ is true in } Q)).$$

Then Γ is true in P .

Proof. By $P \in RM_N^\alpha$ we have $P \in RM_N(RM_N^\xi)$ for any $\xi \in P$ such that $\xi <^\varepsilon \alpha$.

Suppose the Σ_{N+1} -sentence $\varphi := \bigwedge \neg \Gamma := \bigwedge \{\neg \theta : \theta \in \Gamma\}$ is true in P . Then for any $\xi \in P$ with $\xi <^\varepsilon \alpha$ and $x \in P$ there exists a transitive model $Q \in RM_N^\xi \cap P$ of KPi minus Δ_0 -Collection such that $x \in Q$ and φ is true in Q . Let $0 <^\varepsilon \xi <^\varepsilon \alpha$. Then any $Q \in RM_N^\xi$ is Π_N -reflecting for $N \geq 2$, and hence is a model of Δ_0 -Collection. \square

Lemma 3.13 (Elimination of (Ref_{N+1})) *Let $\Gamma \subset \Pi_{N+1}$. Suppose $P_0 \vdash_0^\alpha \Gamma$, $P_0 \in P$ and $P \in RM_N^\alpha$ for a transitive model P of KPi . Then Γ is true in P .*

Proof. This is seen by induction on α . Let $P_0 \vdash_0^\alpha \Gamma$, $P_0 \in P$ and $P \in RM_N^\alpha$ be a transitive model P of KPi . Note that any sentence occurring in the witnessed derivation of $P_0 \vdash_0^\alpha \Gamma$ is Π_{N+1} .

Case 1. First consider the case when the last inference is a (Ref_{N+1}) : By (3) we have $\{\alpha_\ell, \alpha_r\} \subset P_0 \subset P$, $\max\{\alpha_\ell, \alpha_r\} <^\varepsilon \alpha$, $A \in \Pi_{N+1}$.

$$\frac{P_0 \vdash_0^{\alpha_\ell} \Gamma, A(c) \quad P_0 \vdash_0^{\alpha_r} \forall z[ad^z \rightarrow c \in z \rightarrow \neg A^z(c)], \Gamma}{P_0 \vdash_0^\alpha \Gamma} (Ref_{N+1})$$

If $\alpha_\ell = 0$, then there is a $B \in \Gamma \cup A(c)$ such that $B \simeq \bigwedge (B_\iota)_{\iota \in \emptyset}$, i.e., B is either a true Δ_0 -sentence or a sentence $\forall x \in \emptyset C(x)$. In each case we can assume $B \in \Gamma$, and $B^P \in \Gamma^P$ is true.

In what follows assume $0 <^\varepsilon \alpha_\ell$. We can assume that c occurs in $A(c)$, and hence $c \in P_0$.

By Proposition 2.1 we have $P \in RM_N^{\alpha_r}$. From IH we see that

$$\text{either } \forall z \in P[ad^z \rightarrow c \in z \rightarrow \neg A^z(c)] \text{ or } \bigvee \Gamma^P \text{ is true.} \quad (6)$$

On the other hand by IH we have for any $Q \in RM_N^{\alpha_\ell} \cap P$ with $c \in P_0 \in Q \models KPi$ that either $\bigvee \Gamma^Q$ is true or $A(c)^Q$ is true. By (6) for any $Q \in RM_N^{\alpha_\ell} \cap P$ with $P_0 \in Q \models KPi$, $\bigvee \Gamma^Q \vee \bigvee \Gamma^P$ is true. From Proposition 3.12 with $0 <^\varepsilon \alpha_\ell$ we see that $\bigvee \Gamma^P$ is true.

Case 2. Second consider the case when the last inference is a (\bigwedge) : we have $A \simeq \bigwedge (A_\iota)_{\iota \in J}$, $A \in \Gamma$, and $\alpha(\iota) < \alpha$ for any $\iota \in J$

$$\frac{\{P_0(\iota) \vdash_0^{\alpha(\iota)} \Gamma, A_\iota : \iota \in J\}}{P_0 \vdash_0^\alpha \Gamma} (\bigwedge)$$

For any $\iota \in P$ we have $P_0(\iota) \in P$ since P is assumed to be a limit of admissibles.

IH yields for any $\iota \in P$ that either $\bigvee \Gamma^P$ is true or A_ι^P is true. If $J = V$, then we are done. If $J = a \in V$, then $a \in P_0 \subset P$ by (3), and hence $a \subset P$.

Case 3. Third consider the case when the last inference is a (\bigvee) : we have $A \simeq \bigvee (A_\iota)_{\iota \in J}$, $A \in \Gamma$, and $\alpha(\iota) < \alpha$ for an $\iota \in J$

$$\frac{P_0 \vdash_0^{\alpha(\iota)} \Gamma, A_\iota}{P_0 \vdash_0^\alpha \Gamma} (\bigvee)$$

IH yields that either $\bigvee \Gamma^P$ is true or A_ι^P is true. Consider the case when $J = V$. We can assume that ι occurs in A_ι . Then $\iota \in P_0 \subset P$. Hence $\bigvee \Gamma^P$ is true. \square

Let us prove Theorem 2.2. Let $N \geq 2$, and A be a Π_{N+1} -sentence provable in $KPII_{N+1}$. By Embedding 3.9 and Predicative Cut-elimination 3.11 we have for an $m < \omega$, $L_{\omega_1^{CK}} \vdash_0^{\omega_m(\Omega \cdot m)} A$ with $\emptyset = k(A)$, and $L_{\omega_1^{CK}} \vdash_0^{\omega_n(\Omega+1)} A$ for an

$n > m$ with $n < n_0$. If $V \in RM_N^{\omega_n(\Omega+1)}$, then $L_{\omega_1^{CK}} \in V \models \text{KPi}$, and A is true (in V) by Elimination of (Ref_{N+1}) 3.13.

By formalizing the above proof in $\text{FiX}^i(\text{KPi})$ with Lemma 3.1 yields

$$\text{FiX}^i(\text{KPi}) \vdash V \in RM_N([\omega_n(\Omega+1)]; <^\varepsilon) \rightarrow A$$

and then by Theorem 3.2 and $\text{KP}\omega \vdash V \in RM_N([\omega_n(\Omega+1)]; <^\varepsilon) \rightarrow \forall x \exists y [x \in y \wedge ad^y]$

$$\text{KP}\omega \vdash V \in RM_N([\omega_n(\Omega+1)]; <^\varepsilon) \rightarrow A.$$

In the formalization note that we have in $\text{FiX}^i(\text{KPi})$, a partial truth definition of Π_{N+1} -sentences.

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